Estimation of Power Function Distribution Based on Selective Order Statistic

Supplementary material

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Appendix

Proof of Theorem 1.

Let $f^* = \log f_{Y_{j:m}}(y;\ \theta)$, where $f_{Y_{j:m}}(y;\ \theta)$ is defined in (2), then

$$f^* = \log C_{j,m} + \log\theta - \log(1-\theta) + \left(\frac{\theta(j+1)-1}{1-\theta}\right)\log y + (m-j)\log\left(1 - y^{\frac{\theta}{1-\theta}}\right),$$

The derivative of f^* with respect to θ is simplified to

$$\frac{\partial f^*}{\partial\theta} = \frac{1}{\theta(1-\theta)} + \frac{j\log y}{(1-\theta)^2} - \frac{(m-j)y^{\frac{\theta}{1-\theta}}\log y}{(1-\theta)^2\left(1 - y^{\frac{\theta}{1-\theta}}\right)}.$$

The Fisher information about θ contained in $Y_{j:m}$ is

$$I_{Y_{j:m}}(\theta) = E\left(\frac{\partial f^*}{\partial\theta}\right)^2 = C_{j,m}\int_0^1 \left(\frac{1}{\theta(1-\theta)} + \frac{j\log y}{(1-\theta)^2} - \frac{(m-j)y^{\frac{\theta}{1-\theta}}\log y}{(1-\theta)^2(1-y^{\frac{\theta}{1-\theta}})}\right)^2 \times \frac{\theta}{1-\theta} y^{\frac{\theta(j+1)-1}{1-\theta}}\left(1 - y^{\frac{\theta}{1-\theta}}\right)^{m-j} dy.$$

Using the transformation $v = y^{\frac{\theta}{1-\theta}}$, we get

$$I_{Y_{j:m}}(\theta) = C_{j,m}\int_0^1 \left(\frac{1}{\theta(1-\theta)} + \frac{j\log v}{\theta(1-\theta)} - \frac{(m-j)v\log v}{\theta(1-\theta)(1-v)}\right)^2 v^{j-1}(1-v)^{m-j}dv,$$

$$= \frac{C_{j,m}}{\theta^2(1-\theta)^2}\int_0^1 \left(1 + j\log v - \frac{(m-j)v\log v}{(1-v)}\right)^2 v^{j-1}(1-v)^{m-j}dv,$$

$$= \frac{A_{j,m}}{\theta^2(1-\theta)^2}.$$

Proof of Theorem 2. Let $\boldsymbol{y} = \left(Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \dots, Y_{j_0:m}^{(n)}\right)$ represent an iid copies of $Y_{j_0:m}$, then the pdf of $Y_{j_0:m}^{(i)}$ is

$$f_{Y_{j_0:m}^{(i)}}(y;\ \theta) = C_{j_0,m}\frac{\theta}{1-\theta}y^{\frac{\theta(j_0+1)-1}{1-\theta}}\left(1 - y^{\frac{\theta}{1-\theta}}\right)^{m-j_0},$$

where $i = 1, 2, \dots, n$. Also, the log likelihood of $\boldsymbol{y}$ is

$$\log f_{\boldsymbol{Y}}(\boldsymbol{y};\ \theta) = n\log\ C_{j_0,m} + n\log\theta - n\log(1-\theta) + \frac{\theta(j_0+1)-1}{1-\theta}\sum_{i=1}^{n}\log Y_{j_0:m}^{(i)} +$$

$$(m-j_0)\sum_{i=1}^{n}\log\left(1 - \left(Y_{j_0:m}^{(i)}\right)^{\frac{\theta}{1-\theta}}\right).$$

Taking the first derivative of $\log f_{\boldsymbol{Y}}(\boldsymbol{y};\ \theta)$ with respect to θ, we get

$$\frac{\partial\log f_{\boldsymbol{Y}}(\boldsymbol{y};\ \theta)}{\partial\theta} = \frac{n}{\theta} + \frac{n}{(1-\theta)} + \frac{j_0\sum_{i=1}^{n}\log Y_{j_0:m}^{(i)}}{(1-\theta)^2} - (m-j_0)\sum_{i=1}^{n}\frac{\left(Y_{j_0:m}^{(i)}\right)^{\frac{\theta}{1-\theta}}\log\left(Y_{j_0:m}^{(i)}\right)}{(1-\theta)^2\left(1-\left(Y_{j_0:m}^{(i)}\right)^{\frac{\theta}{1-\theta}}\right)}.$$

Setting the last equation equal zero and solving for θ yields the following

$$(m-j_0)\sum_{i=1}^{n}\frac{\left(Y_{j_0:m}^{(i)}\right)^{\frac{\theta}{1-\theta}}\log\left(Y_{j_0:m}^{(i)}\right)}{1-\left(Y_{j_0:m}^{(i)}\right)^{\frac{\theta}{1-\theta}}} - j_0\sum_{i=1}^{n}\log Y_{j_0:m}^{(i)} = \frac{n(1-\theta)}{\theta}. \qquad (3)$$

The solution of equation (3) represents the MLE of θ which is denoted by $\hat{\theta}_{MLE}$ provided that $\frac{\partial^2\log f_{\boldsymbol{Y}}(\boldsymbol{y};\theta)}{\partial\theta^2}|_{\theta=\hat{\theta}_{MLE}} < 0$. It can be shown that the pdf (2) satisfies the conditions of Theorem 4.1 (page 429) of Lehmann and Casella (1983). Hence with probability tending to 1, there exists a solution for equation (3) with respect to θ allowing us to conclude the consistency and the asymptotic normality of the MLE.

To prove the last part of the theorem, we define $\vartheta = \theta/(1-\theta)$ which implies that $\vartheta \in (0,\infty)$. It can be noticed that for $\vartheta > 0$, equation (3) is equivalent to

$$\vartheta(m-j_0)\sum_{i=1}^{n}\frac{\left(Y_{j_0:m}^{(i)}\right)^{\vartheta}\log\left(Y_{j_0:m}^{(i)}\right)}{1-\left(Y_{j_0:m}^{(i)}\right)^{\vartheta}} - j_0\vartheta\sum_{i=1}^{n}\log Y_{j_0:m}^{(i)} - n = 0. \quad (4)$$

Let $q(\vartheta)$ be the left hand side of (4), then we need to show that $q'(\vartheta)$ is positive for every $\vartheta \in (0,\infty)$. Notice that

$$q'(\vartheta) = \vartheta(m-j_0)\sum_{i=1}^{n}\frac{\left(Y_{j_0:m}^{(i)}\right)^{\vartheta}\left(\log\left(Y_{j_0:m}^{(i)}\right)\right)^2}{\left(1-\left(Y_{j_0:m}^{(i)}\right)^{\vartheta}\right)^2} + (m-j_0)\sum_{i=1}^{n}\frac{\left(Y_{j_0:m}^{(i)}\right)^{\vartheta}\log\left(Y_{j_0:m}^{(i)}\right)}{1-\left(Y_{j_0:m}^{(i)}\right)^{\vartheta}}$$
$$- j_0\sum_{i=1}^{n}\log Y_{j_0:m}^{(i)}.$$

Also

$$q'(\vartheta) \geq \vartheta(m-j_0)\sum_{i=1}^{n}\frac{\left(Y_{j_0:m}^{(i)}\right)^{\vartheta}\left(\log\left(Y_{j_0:m}^{(i)}\right)\right)^2}{1-\left(Y_{j_0:m}^{(i)}\right)^{\vartheta}} + (m-j_0)\sum_{i=1}^{n}\frac{\left(Y_{j_0:m}^{(i)}\right)^{\vartheta}\log\left(Y_{j_0:m}^{(i)}\right)}{1-\left(Y_{j_0:m}^{(i)}\right)^{\vartheta}}$$
$$- j_0\sum_{i=1}^{n}\log Y_{j_0:m}^{(i)}. \quad (5)$$

Since the third term on the right hand side of (5) is positive, it suffices to show that the sum of the remaining terms is positive. Notice that $a(\log x)^2 + \log x \geq 0$, for every $a > 0, x > 0$ allows us to conclude that $q'(\vartheta) > 0$ for every ϑ. Also $q(0) < 0$ and $\lim_{\vartheta \to 1^-} q(\vartheta) > 0$ show that there exist a unique value of $\vartheta_0 \in (0,\infty)$ such that $q(\vartheta_0) = 0$. Since $\vartheta = \theta/(1-\theta)$ is a one-to-one transformation, then there exists a unique solution of (3) that falls inside $(0,1)$.

Proof Theorem 3.

Let $I_1(\theta)$ and $I_2(\theta)$ denote the Fisher information numbers obtained using the following two independent SOS samples $\boldsymbol{x} = \left(X_{j_0:m}^{(1)}, X_{j_0:m}^{(2)}, \dots, X_{j_0:m}^{(n)}\right)$ and $\boldsymbol{y} = \left(Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \dots, Y_{j_0:m}^{(n)}\right)$, respectively. Since $\sqrt{n}\left(\hat{\theta}_{1,MLE}-\theta_1\right)\xrightarrow{D}N(0,\ I_1^{-1}(\theta_1))$ and $\sqrt{n}\left(\hat{\theta}_{2,MLE}-\theta_2\right)\xrightarrow{D}N(0,\ I_2^{-1}(\theta_2))$, independently, then

$$\sqrt{n}\left(\begin{bmatrix}\hat{\theta}_{1,MLE}\\ \hat{\theta}_{2,MLE}\end{bmatrix}-\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}\right)\xrightarrow{D}N_2\left(\mathbf{0},\quad \begin{bmatrix}I_1^{-1}(\theta_1) & 0\\ 0 & I_2^{-1}(\theta_2)\end{bmatrix}\right).$$

We apply the Delta method for the MLE of $\mathcal{R}$ by setting $\hat{\mathcal{R}}_{MLE}=g\left(\hat{\theta}_{1,MLE},\hat{\theta}_{2,MLE}\right)$, where

$$g(\theta_1,\theta_2)=\frac{\theta_2(1-\theta_1)}{\theta_1+\theta_2-2\theta_1\theta_2}.$$

So

$$\nabla^T=\left[\frac{\partial g}{\partial\theta_1},\frac{\partial g}{\partial\theta_2}\right]=\left[\frac{-\theta_2(1-\theta_2)}{(\theta_1+\theta_2-2\theta_1\theta_2)^2},\frac{\theta_1(1-\theta_1)}{(\theta_1+\theta_2-2\theta_1\theta_2)^2}\right]$$

The variance of $\hat{\mathcal{R}}_{MLE}$ is

$$\nabla^T\begin{bmatrix}I_1^{-1}(\theta_1) & 0\\ 0 & I_2^{-1}(\theta_2)\end{bmatrix}\nabla=\frac{2\theta_1^2\theta_2^2(1-\theta_1)^2(1-\theta_2)^2}{A_{j,m}(\theta_1+\theta_2-2\theta_1\theta_2)^4}.$$

Since $\hat{\theta}_{1,MLE}\xrightarrow{P}\theta_1$ and $\hat{\theta}_{2,MLE}\xrightarrow{P}\theta_2$, then using the continuity of the Fisher information number and the function g, the following asymptotic convergence remains valid:

$$\sqrt{n}I_1^{-1/2}(\hat{\theta}_{1,MLE})\left(\hat{\theta}_{1,MLE}-\theta_1\right)\xrightarrow{D}N(0,1)$$

and

$$\sqrt{n}I_2^{-1/2}(\hat{\theta}_{2,MLE})\left(\hat{\theta}_{2,MLE}-\theta_2\right)\xrightarrow{D}N(0,1),$$

Hence

$$\widehat{\sigma^2}_{\hat{\mathcal{R}}_{MLE}}=\frac{2\hat{\theta}_{1,MLE}^2\hat{\theta}_{2,MLE}^2\left(1-\hat{\theta}_{1,MLE}\right)^2\left(1-\hat{\theta}_{2,MLE}\right)^2}{A_{j,m}\left(\hat{\theta}_{1,MLE}+\hat{\theta}_{2,MLE}-2\hat{\theta}_{1,MLE}\hat{\theta}_{2,MLE}\right)^2}$$

Thus, we get $\sqrt{n}\sqrt{\widehat{\sigma^2}_{\hat{\mathcal{R}}_{MLE}}}\left(\hat{\mathcal{R}}_{MLE}-\mathcal{R}\right)\xrightarrow{D}N(0,1)$

Proof of Theorem 4. Let $Y_{j_0:m}^{(1)},\dots,Y_{j_0:m}^{(n)}$ be an SOS sample from $PFD(\theta)$ where $j_0=1$ or 2 and $m\le 10$. Then the MOME of θ is obtained by solving the following equation for θ:

$$\bar{Y}_{j_0:m}=\frac{m!\,\theta^m}{\prod_{i=0}^{m-1}(i\theta+1)},\qquad \text{if}\ \ j_0=1$$

or the equation

$$\bar{Y}_{j_0:m} = \frac{m!\,\theta^m}{\prod_{i=0}^{m-1}(i\theta+1)}, \qquad \text{if} \;\; j_0 = 2.$$

Notice that the equation $\bar{Y}_{j:m} = \frac{m!\theta^m}{\prod_{i=0}^{m-1}(i\theta+1)}$ is equivalent to the polynomial equation

$$m!\,\theta^m - \bar{Y}_{j_0:m}\prod_{i=0}^{m-1}(i\theta+1) = 0.$$

Applying the Descartes' theorem on polynomial equations (Kurosh, 1972) which states that the number of positive roots of a polynomial equals the number of sign changes in its coefficients, we find that the number of sign changes in its coefficients is equal to 1. Hence the number of positive roots for the equation $m!\,\theta^m - \bar{Y}_{j_0:m}\prod_{i=0}^{m-1}(i\theta+1) = 0$ is only one positive root. To prove that this root belongs to $(0,1)$, we notice that the function $h(\theta) = m!\,\theta^m - \bar{Y}_{j_0:m}\prod_{i=0}^{m-1}(i\theta+1)$, $\theta \in [0,1]$ is continuous and satisfies $h(0) < 0$ and $h(1) = m!\,(1 - \bar{Y}_{j_0:m}) > 0$. Hence, Bolzano's theorem guarantees that $h(\theta)$ has at least one real root between 0 and 1. Since $h(\theta)$ has only one positive root, then it must be a unique root between 0 and 1. Also we conclude that the derivative of the inverse function is strictly monotone.

Proof of Theorem 5.

Obtaining the asymptotic distribution of $\hat{\mathcal{R}}_{MOM}$ can be achieved using the asymptotic distributions of $\hat{\theta}_{1,MOM}$ and $\hat{\theta}_{2,MOM}$. To do so, we define $K(\theta)$ to be the inverse function of $h(\theta)$. Notice that $h(\theta)$ is differentiable which implies that $K(\theta)$ is differentiable too. Hence, the estimators $\hat{\theta}_{1,MOM}$ and $\hat{\theta}_{2,MOM}$ can be expressed in terms of $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{y}}$ as follows: $\hat{\theta}_{1,MOM} = K(\bar{\boldsymbol{x}})$ and $\hat{\theta}_{2,MOM} = K(\bar{\boldsymbol{y}})$. So the central limit theorem concludes that (show that $\hat{\theta}_{1,MOM} \xrightarrow{P} \theta_1$)

$$\sqrt{n}\left(\bar{\boldsymbol{x}} - \frac{m!\,\theta_1^m}{\prod_{i=0}^{m-1}(i\theta_1+1)}\right) \xrightarrow{d} N\left(0,\;\; \sigma^2_{1,X_{j:m}}\right)$$

and

$$\sqrt{n}\left(\bar{\boldsymbol{y}} - \frac{m!\,\theta_2^{m-1}}{\prod_{i=0}^{m-1}(i\theta_2+1)}\right) \xrightarrow{d} N\left(0,\;\; \sigma^2_{2,Y_{j:m}}\right),$$

where $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{y}}$ are the sample means of $\left(X^{(1)}_{j_0:m}, X^{(2)}_{j_0:m}, \dots, X^{(n)}_{j_0:m}\right)$ and $\left(Y^{(1)}_{j_0:m}, Y^{(2)}_{j_0:m}, \dots, Y^{(n)}_{j_0:m}\right)$, respectively, and $\sigma^2_{i,Y_{j:m}}$ is the variance of the pdf (2), when $\theta = \theta_i$, $i = 1,\ 2$. Now the delta method implies that

$$\sqrt{n}\left(\hat{\theta}_{1,MOM} - \theta_1\right) \xrightarrow{d} N\left(0, \sigma^2_{1,X_{j_0:m}} {K'}^2(\theta_1)\right)$$

and

$$\sqrt{n}\left(\hat{\theta}_{2,MOM} - \theta_2\right) \xrightarrow{d} N\left(0, \sigma^2_{2,Y_{j_0:m}} {K'}^2(\theta_2)\right)$$

Apply the continuous mapping theorem for convergence in probability and reapplying the delta method, we get $V\left(\hat{\theta}_{i,MOM}\right) \xrightarrow{d} V(\theta_i)$, and $K'\left(\hat{\theta}_{i,MOM}\right) \xrightarrow{d} K(\theta_i)$, for $i = 1, 2$.

Now we intend to derive the asymptotic distribution of the $\hat{\mathcal{R}}_{MOM}$

$$\hat{\mathcal{R}}_{MOM} = \frac{\hat{\theta}_{2,MOM}\left(1 - \hat{\theta}_{1,MOM}\right)}{\hat{\theta}_{1,MOM} + \hat{\theta}_{2,MOM} - 2\hat{\theta}_{1,MOM}\hat{\theta}_{2,MOM}} = g\left(\hat{\theta}_{1,MOM}, \hat{\theta}_{2,MOM}\right).$$

Let $V(\theta_i) = \sigma^2_{i,Y_{j_0:m}}$, for $i = 1, 2$ and following the same argument as in the proof of Theorem 3, we get

$$\sqrt{n}\left(\begin{bmatrix}\hat{\theta}_{1,MOM} \\ \hat{\theta}_{2,MOM}\end{bmatrix} - \begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix}\right) \xrightarrow{D} N_2\left(\mathbf{0}, \quad \begin{bmatrix} V(\theta_1){K'}^2(\theta_1) & 0 \\ 0 & V(\theta_2){K'}^2(\theta_2)\end{bmatrix}\right),$$

therefore

$$\mathcal{T}^2(\theta_1, \theta_2) = \nabla^T \begin{bmatrix} V(\theta_1){K'}^2(\theta_1) & 0 \\ 0 & V(\theta_2){K'}^2(\theta_2)\end{bmatrix} \nabla$$

$$= \frac{V(\theta_1){K'}^2(\theta_1)\theta_2^2(1-\theta_2)^2 + V(\theta_2){K'}^2(\theta_2)\theta_1^2(1-\theta_1)^2}{(\theta_1 + \theta_2 - 2\theta_1\theta_2)^4}$$

Table 4. The bias of the MLE and MOM estimators when $\theta_1 = 0.3$ and $\theta_2 = 0.6$ for different values of m and n.

n	m	$\hat{\theta}_{1,MLE}$	$\hat{\theta}_{2,MLE}$	$\hat{\theta}_{1,MOM}$	$\hat{\theta}_{2,MOM}$	$\hat{R}_{MLE}$	$\hat{R}_{MOM}$	$\hat{\psi}_{MLE}$	$\hat{\psi}_{MOM}$
10	1	.0145	.0101	-.0520	.0723	-.0090	-.0063	.0431	.0553
	3	.0040	.0013	-.0041	-.0034	-.0042	.0045	.0185	.0191
	5	.0021	.0012	-.0031	-.0025	-.0032	.0038	.0151	.0189
	6	.0020	-.0011	.0028	-.0024	.0031	.0035	-.0143	.0172
	8	.0015	.0010	-.0025	-.0023	-.0026	-.0032	.0011	-.0165
	10	.0009	.0008	-.0020	.0021	-.0025	-.0030	.0009	-.0011
20	1	.0079	.0053	.0431	.0062	-.0048	-.0027	.0148	.0401

	3	.0026	.0009	-.0029	-.0027	-.0023	.0022	.0076	.0162
	5	.0009	.0003	-.0023	-.0022	.0020	.0020	.0057	.0142
	6	.0015	-.0002	-.0022	.0021	.0019	-.0019	-.0054	.0135
	8	-.0012	-.0001	-.0014	-.0020	-.0019	-.0019	-.0008	-.0127
	10	-.0008	.0000	.0010	-.0017	-.0018	-.0015	.0008	.0099
30	1	.0038	.0033	-.0351	.0053	-.0039	-.0021	.0125	.0317
	3	.0012	.0009	-.0027	-.0021	-.0021	.0020	.0070	.0231
	5	.0008	.0003	-.0023	-.0018	-.0018	.0019	.0049	.0126
	6	.0008	-.0002	.0022	-.0015	.0016	.0017	-.0044	-.0114
	8	-.0007	-.0001	-.0019	-.0011	-.0013	.0016	.0007	-.0103
	10	-.0006	.0001	-.0010	.0010	-.0010	-.0013	-.0005	-.0008
40	1	.0036	.0021	-.0247	.0034	-.0035	-.0017	.0068	.0294
	3	.0010	.0007	-.0018	-.0028	-.0031	.0013	.0033	.0236
	5	.0006	.0002	-.0021	.0013	-.0029	.0012	.0027	.0108
	6	-.0005	-.0002	-.0019	-.0011	-.0022	-.0011	-.0023	.0101
	8	-.0004	-.0001	.0010	.0010	-.0009	.0009	-.0021	.0009
	10	-.0004	.0001	-.0010	.0008	.0009	-.0008	.0014	-.0008
50	1	.0027	.0011	.0202	-.0027	-.0024	-.0013	.0010	.0254
	3	.0001	.0005	-.0012	-.0024	-.0020	.0010	.0009	.0201
	5	.0004	.0001	-.0010	-.0010	-.0019	.0010	.0009	.0008
	6	.0003	-.0001	-.0009	-.0009	-.0017	-.0009	.0008	.0007
	8	-.0003	-.0001	-.0009	-.0009	-.0008	.0008	.0005	-.0006
	10	-.0002	.0001	.0008	.0007	.0006	.0005	-.0003	.0005

Table 5. The MSE of the MLE and MOM estimators when $\theta_1 = 0.3$ and $\theta_2 = 0.6$ for different values of m and n.

n	m	$\hat{\theta}_{1,MLE}$	$\hat{\theta}_{2,MLE}$	$\hat{\theta}_{1,MOM}$	$\hat{\theta}_{2,MOM}$	$\hat{R}_{MLE}$	$\hat{R}_{MOM}$	$\hat{\psi}_{MLE}$	$\hat{\psi}_{MOM}$
10	1	.0051	.0056	.0085	.0067	.0066	.0096	.2715	1.0298
	3	.0018	.0023	.0039	.0027	.0025	.0040	.1053	.3282
	5	.0012	.0016	.0027	.0018	.0017	.0027	.0737	.2151
	6	.0011	.0015	.0023	.0017	.0015	.0023	.0639	.1985
	8	.0009	.0012	.0020	.0014	.0012	.0021	.0582	.1512
	10	.0008	.0010	.0019	.0013	.0011	.0019	.0216	.1219
20	1	.0024	.0029	.0024	.0035	.0031	.0048	.1330	.3344
	3	.0009	.0011	.0020	.0013	.0012	.0019	.0514	.1295
	5	.0007	.0008	.0014	.0009	.0008	.0014	.0346	.0842
	6	.0006	.0006	.0011	.0009	.0006	.0013	.0261	.0781
	8	.0005	.0002	.0011	.0008	.0004	.0011	.0151	.0617
	10	.0005	.0001	.0009	.0007	.0004	.0010	.0201	.0311
30	1	.0015	.0019	.0029	.0023	.0020	.0032	.0890	.2050
	3	.0005	.0008	.0012	.0009	.0008	.0013	.0342	.0766
	5	.0004	.0005	.0009	.0006	.0006	.0009	.0192	.0527
	6	.0004	.0004	.0009	.0006	.0006	.0008	.0181	.0495
	8	.0003	.0002	.0008	.0004	.0005	.0006	.0119	.0433

	10	.0001	.0000	.0007	.0002	.0005	.0004	.0091	.0222
40	1	.0013	.0014	.0021	.0017	.0015	.0023	.0639	.1350
	3	.0005	.0006	.0010	.0007	.0006	.0010	.0256	.0574
	5	.0003	.0004	.0007	.0004	.0004	.0007	.0179	.0383
	6	.0003	.0003	.0007	.0004	.0004	.0007	.0167	.349
	8	.0001	.0003	.0006	.0003	.0003	.0005	.0094	.0241
	10	.0001	.0003	.0005	.0003	.0003	.0005	.0083	.0215
50	1	.0009	.0011	.0017	.0014	.0012	.0019	.0524	.1058
	3	.0003	.0005	.0007	.0005	.0005	.0008	.0202	.0414
	5	.0003	.0003	.0006	.0004	.0004	.0006	.0170	.0318
	6	.0002	.0003	.0006	.0004	.0003	.0005	.0166	.0299
	8	.0001	.0002	.0005	.0003	.0003	.0005	.0010	.0269
	10	.0001	.0001	.0003	.0002	.0003	.0004	.0081	.0227

Table 6.The expected lengths of the confidence intervals when $\theta = 0.3$ and 0.6 for different values of m and n.

j_0	m	$\theta = 0.3$					$\theta = 0.6$				
		n					n				
		10	20	30	40	50	10	20	30	40	50
1	1	0.2488	0.1773	0.1459	0.1263	0.1134	0.4073	0.2958	0.2436	0.2116	0.1897
	2	0.1966	0.1397	0.1147	0.0999	0.0891	0.2977	0.2130	0.1744	0.1514	0.1357
	3	0.1349	0.0969	0.0798	0.0692	0.0620	0.1788	0.1312	0.1082	0.0942	0.0846
	4	0.1019	0.0737	0.0606	0.0526	0.0471	0.1273	0.0930	0.0768	0.0669	0.0601
	5	0.0843	0.0608	0.0499	0.0432	0.0387	0.1016	0.0740	0.0699	0.0531	0.0476
2	6	0.0566	0.0406	0.0335	0.0291	0.0261	0.0585	0.0427	0.0352	0.0306	0.0274
	7	0.0494	0.0354	0.0292	0.0253	0.0227	0.0507	0.0367	0.0302	0.0263	0.0236
	8	0.0445	0.0319	0.0262	0.0227	0.0204	0.0454	0.0328	0.0269	0.0235	0.0210
	9	0.0409	0.0293	0.0240	0.0209	0.0187	0.0416	0.0310	0.0247	0.0215	0.0192
	10	0.0382	0.0274	0.0224	0.0195	0.0174	0.0388	0.0280	0.0230	0.0200	0.0179